

5.175 k_alldifferent

	DESCRIPTION	LINKS	GRAPH
Origin	[133]		
Constraint	<code>k_alldifferent(VARS)</code>		
Synonyms	<code>k_alldiff</code> , <code>k_alldistinct</code> , <code>some_different</code> .		
Type	<code>X</code> : <code>collection(x-dvar)</code>		
Argument	<code>VARS</code> : <code>collection(vars - X)</code>		
Restrictions	<code>required(X, x)</code> <code>required(VARS, vars)</code> $ X > 0$ $ VARS > 0$		
Purpose	For each collection of variables depicted by an item of VARS, enforce their corresponding variables to take distinct values.		
Example	$\left(\left\langle \text{vars} - \left\langle \begin{array}{c} x - 5, \\ x - 6, \\ x - 0, \\ x - 9, \\ x - 3 \end{array} \right\rangle, \right\rangle \right)$ $\text{vars} - \langle 5, 6, 1, 2 \rangle$		
	The <code>k_alldifferent</code> constraint holds since all the values 5, 6, 0, 9 and 3 are distinct and since all the values 5, 6, 1 and 2 are distinct as well.		
Typical	$ X > 1$ $ VARS > 1$		
Symmetries	<ul style="list-style-type: none"> • Items of VARS are permutable. • Items of VARS.vars are permutable. • All occurrences of two distinct values of VARS.vars.x can be swapped; all occurrences of a value of VARS.vars.x can be renamed to any unused value. 		
Usage	Systems of <code>alldifferent</code> constraints sharing variables occurs frequently in practice. We give 4 typical problems that can be modelled by a combination of <code>alldifferent</code> constraints as well as one problem where a system of <code>alldifferent</code> constraints provides a necessary condition.		

- The *graph colouring* problem is to colour with a restricted number of colours the vertices of a given undirected graph in such a way that adjacent vertices are coloured with distinct colours. The problem can be modelled by a system of **alldifferent** constraints. All the next problems can be seen as graph colouring problems where the graphs have some specific structure.
- A *Latin square of order n* is an $n \times n$ array in which n distinct numbers in $[1, n]$ are arranged so that each number occurs once in each row and column. The problem is to complete a partially filled Latin square. Part (A) of Figure 5.354 gives a partially filled Latin square, while part (B) provides a possible completion.

1			
			3
3			
			1

(A)

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(B)

Figure 5.354: A partially filled Latin square and a possible completion

- A *Sudoku* is a Latin square of order 9×9 such that the numbers in each major 3×3 block are distinct. As for the Latin square problem, the problem is to complete a partially filled board. Part (A) of Figure 5.355 gives a partially filled Sudoku board, while part (B) provides a possible completion. A constraint programming approach for solving Sudoku puzzles is depicted in [350]. It shows how to generate redundant constraints as well as shaving [247] in order to find a solution without guessing.

	2	6				8	1	
3			7		8			6
4				5				7
	5		1		7		9	
		3	9		5	1		
	4		3		2		5	
1				3				2
5			2		4			9
	3	8				4	6	

(A)

7	2	6	4	9	3	8	1	5
3	1	5	7	2	8	9	4	6
4	8	9	6	5	1	2	3	7
8	5	2	1	4	7	6	9	3
6	7	3	9	8	5	1	2	4
9	4	1	3	6	2	7	5	8
1	9	4	8	3	6	5	7	2
5	6	7	2	1	4	3	8	9
2	3	8	5	7	9	4	6	1

(B)

Figure 5.355: A partially Sudoku square and a possible completion

- A *task assignment* problem consists to assign a given set of non-preemptive tasks, which are fixed in time (i.e., the origin, duration and end of each task are fixed), to a set of resources so that, tasks that are assigned to the same resource do not overlap in time. Each task can be assigned to a predefined set of resources. Problems like *aircraft stand allocation* [124], [349] or *air traffic flow management* [19] correspond to an example of a real-life task **assignment** problem. *Assignment of service professionals* [12] is yet another industrial example where professionals have to be

assigned positions in such a way that positions assigned to a given professional do not overlap in time.

Part (A) of Figure 5.356 gives an example of task assignment problem. For each task we indicate the set of resources where it can potentially be assigned (i.e., the domain of its assignment variable). For instance, task T1 can be assigned to resources 1 or 2. Part (B) of Figure 5.356 gives the corresponding interval graph: We have one vertex for each task and an edge between two tasks that overlap in time. We have a system of alldifferent constraints corresponding to the maximum cliques of the interval graph (i.e., {T1,T5,T8}, {T2,T5,T8}, {T2,T6}, {T3,T6,T9}, {T3,T7,T9}, {T4,T7,T9}). Finally, part (C) of Figure 5.356 provides a possible solution to the task assignment problem where tasks T1, T2, T9 are assigned to resource 1, tasks T3, T4, T8 are assigned to resource 2, and tasks T5, T6, T7 are assigned to resource 3.

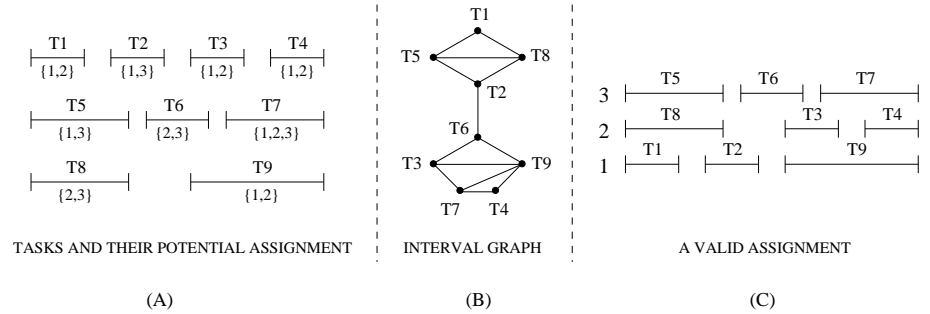


Figure 5.356: A task assignment problem, its corresponding interval graph and a possible solution

- The *tree partitioning with precedences* problem is to compute a vertex-partitioning of a given digraph \mathcal{G} in disjoint trees (i.e., a forest), so that a given set of precedences holds. The problem can be modelled with a `tree_precedence(NTREE, VERTICES)` constraint, where `NTREE` is a domain variable specifying the numbers of trees in the forest and `VERTICES` is a collection of the digraph's n vertices. Each item $v \in \text{VERTICES}$ has the following attributes, which complete the description of the digraph:

- `index` is an integer in $[1, n]$ that can be interpreted as the *label* of v .
- `father` is a domain variable whose domain consists of elements (vertex label) of $[1, n]$. It can be interpreted as the *unique successor* of v .
- `preds` is a possibly empty set of integers, its elements (vertex label) being in $[1, n]$. It can be interpreted as the *mandatory ancestors* of v .

We model the `tree_precedence` constraint by the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which the vertices represent the elements of `VERTICES` and the arcs represent the successors relations between them. Formally, \mathcal{G} is defined as follows:

- To the i^{th} vertex ($1 \leq i \leq n$), `VERTICES`[i], of the `VERTICES` collection corresponds a vertex of \mathcal{V} denoted by v_i .
- For every pair of vertices (`VERTICES`[i], `VERTICES`[j]), where i and j are not necessarily distinct, there is an arc from v_i to v_j in \mathcal{E} .

The `tree_precedence` constraint specifies that its associated digraph \mathcal{G} should be a forest that fulfils the precedence constraints. Formally a ground instance of a `tree_precedence(NTREE, VERTICES)` constraint is satisfied if and only if the following conditions hold:

1. $\forall i \in [1, n] : \text{VERTICES}[i].\text{index} = i,$
2. Its associated digraph \mathcal{G} consists of `NTREE` connected components,
3. Each connected component of \mathcal{G} does not contain any circuit involving more than one vertex,
4. For every vertex `VERTICES`[i] such that $j \in \text{VERTICES}[i].\text{preds}$ there must be an elementary path in \mathcal{G} from `VERTICES`[j] to `VERTICES`[i].

We can build the following system of `alldifferent` constraints that corresponds to a necessary condition for the `tree_precedence` constraint: To each vertex v of \mathcal{G} , which both has no predecessors and cannot be the root of a tree, we generate an `alldifferent` constraint involving the father variables of those descendants of v in \mathcal{G} that cannot be the root of a tree.

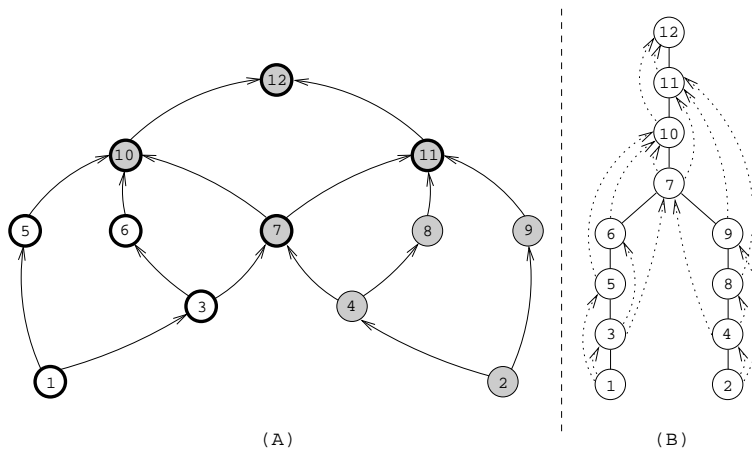


Figure 5.357: A set of precedences and a corresponding feasible tree

For the set of precedences depicted by part (A) of Figure 5.357⁹, where we assume that `VERTICES`[12] is the only vertex that can be a root and where F_i denotes the father variable associated with `VERTICES`[i], we get the following system of `alldifferent` constraints:

- `alldifferent`($\langle F_1, F_3, F_5, F_6, F_7, F_{10}, F_{11} \rangle$),
- `alldifferent`($\langle F_2, F_4, F_7, F_8, F_9, F_{10}, F_{11} \rangle$).

The variables of these two `alldifferent` constraints respectively correspond to the descendants of the two source vertices (i.e., F_1 and F_2) of the precedence graph depicted by part (A) of Figure 5.357. On part (A) of Figure 5.357 the descendants of F_1 and F_2 are respectively depicted with a thick line and a grey circle. Their intersection, $\{F_7, F_{10}, F_{11}, F_{12}\}$, from which we remove

⁹The number in a vertex gives the value of the `index` attribute of the corresponding item.

F_{12} belong to the two `alldifferent` constraints. In fact, F_{12} is not mentioned in the two `alldifferent` constraints since its corresponding vertex is the root of a tree. Part (B) of Figure 5.357 gives a possible tree satisfying all the precedences constraints expressed by part (A), where precedences are depicted with a dotted line. It corresponds to the following ground solution:

```
tree_precedence((
  index - 1   father - 3   preds - {},
  index - 2   father - 4   preds - {},
  index - 3   father - 5   preds - {1},
  index - 4   father - 8   preds - {2},
  index - 5   father - 6   preds - {1},
  index - 6   father - 7   preds - {3},
  index - 7   father - 10  preds - {3, 4},
  index - 8   father - 9   preds - {4},
  index - 9   father - 7   preds - {2},
  index - 10  father - 11  preds - {5, 6, 7},
  index - 11  father - 12  preds - {7, 8, 9},
  index - 12  father - 12  preds - {10, 11} ))
```

Remark

It was shown in [134] that, finding out whether a system of two `alldifferent` constraints sharing some variables has a solution or not is NP-hard. This was achieved by reduction from `set packing`.

A slight variation in the way of describing the arguments of the `k_alldifferent` constraint appears in [324] under the name of `some_different`: the set of disequalities is described by a set of pairs of variables, where each pair corresponds to a disequality constraint between two given variables.

Within the context of `linear programming`, a relaxation of the `k_alldifferent` constraint is provided in [7]. The special case where $k = 2$ is discussed in [8].

Algorithm

Even if there is no filtering algorithm for the `k_alldifferent` constraint, one can enforce redundant constraints for the following patterns:

- Within the context of graph colouring, one can state an `nvalue` constraint for every cycle of odd length of the graph to colour enforcing that the corresponding variables have to be assigned to at least three distinct values.
- Within the context of Latin squares, one can state a `colored_matrix` constraint enforcing that each value is used exactly once in each row and column.
- Within the context of two `alldifferent` constraints `alldifferent`(($U_1, \dots, U_n, V_1, \dots, V_m$)) and `alldifferent`(($U_1, \dots, U_n, W_1, \dots, W_m$)) where the domain of all variables $U_1, \dots, U_n, V_1, \dots, V_m, W_1, \dots, W_m$ is included in the interval $[1, n + m]$, one can state a `same_and_global_cardinality` constraint stating that the variables V_1, \dots, V_m should correspond to a permutation of the variables W_1, \dots, W_m and that the variables V_1, \dots, V_m should be assigned to distinct values.
- In the general case of two `alldifferent` constraints `alldifferent`(($U_1, \dots, U_n, V_1, \dots, V_m$)) and `alldifferent`(($U_1, \dots, U_n, W_1, \dots, W_o$)), one can state an `nvalue` constraint involving the variables V_1, \dots, V_m and W_1, \dots, W_o enforcing that these variables should not use more than $s - n$ distinct values, where s denotes the cardinality of the union of the domains of the variables $U_1, \dots, U_n, V_1, \dots, V_m, W_1, \dots, W_o$.

Several propagation rules for the `k_alldifferent` constraint are also described in [226].

Reformulation

Given two `alldifferent` constraints that share some variables, a reformulation preserving `bound-consistency` was introduced in [67]. This reformulation is based on an extension of Hall's theorem that is presented in the same paper.

See also

common keyword: `colored_matrix` (*system of constraints*).

generalisation: `diffn`, `geost` (*tasks for which the start attribute is not fixed*).

part of system of constraints: `alldifferent`.

related: `nvalue` (*implied by two overlapping `alldifferent`*), `same_and_global_cardinality` (*implied by two overlapping `alldifferent` and restriction on values*).

Keywords

application area: air traffic management, assignment.

characteristic of a constraint: all different, disequality.

combinatorial object: permutation, Latin square.

complexity: set packing.

constraint type: system of constraints, overlapping `alldifferent`, value constraint, decomposition.

filtering: bound-consistency, duplicated variables.

problems: graph colouring.

puzzles: Sudoku.

	For all items of VARS:
Arc input(s)	<code>VARS.vars</code>
Arc generator	<code>CLIQUE</code> \mapsto <code>collection(x1, x2)</code>
Arc arity	2
Arc constraint(s)	<code>x1.x = x2.x</code>
Graph property(ies)	<u><code>MAX_NSCC</code> \leq 1</u>

Graph model

For each collection of variables depicted by an item of VARS we generate a *clique* with an *equality* constraint between each pair of vertices (including a vertex and itself) and state that the size of the largest strongly connected component should not exceed one.

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