Generalized Interval Projection: A New Technique for Consistent Domain Extension

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Abstract

This paper deals with systems of parametric equations over the reals, in the framework of interval constraint programming. As parameters vary within intervals, the solution set of a problem may have a non null volume. In these cases, an inner box (i.e., a box included in the solution set) instead of a single punctual solution is of particular interest, because it gives greater freedom for choosing a solution. Our approach is able to build an inner box for the problem starting with a single point solution, by consistently extending the domain of every variable. The key point is a new method called *generalized projection*.

The requirements are that each parameter must occur only once in the system, variable domains must be bounded, and each variable must occur only once in each constraint. Our extension is based on an extended algebraic structure of intervals called generalized intervals, where improper intervals are allowed (e.g. [1,0]).

1 Introduction

The purpose of this paper will be illustrated on a simple example of signal relay positioning.

The situation is as follows. m units are deployed on an area, each of them being equipped with a transceiver. Because of the limited power of their transceivers, the units cannot communicate. The question is to position a relay such that all units get connected.

We denote (a_i, b_i) the coordinates of the i^{th} unit position, and d_i its distance from the relay. Assume first that all a_i, b_i and d_i are fixed. Then, the model consists in m simple distance equations and is easily solved by any traditional algebraic or numerical technique. Since the system is probably unfeasible, a least-square method can provide a point making each distance being as close as possible to the desired value d_i.

Unfortunately, this model suffers from three serious limitations:

- Distances should not be fixed. The distance d_i must be neither more than the transceiver range $\overline{\mathbf{d}_i}$, nor less than a lower bound $\underline{\mathbf{d}_i}$, say, because of the damaging loop effect. Hence, distances must rather be assigned intervals $\mathbf{d}_1$, ...,$\mathbf{d}_m$.
- Positions of units are not fixed neither. They usually patrol around their position and can move in a box $\mathbf{a}_i \times \mathbf{b}_i$ to pick up the signal.
- Providing a single solution (x, y) is often not realistic. E.g., an antenna cannot be installed exactly at a precise position in presence of obstacles. Therefore, one is rather interested by a box $\mathbf{x} \times \mathbf{y}$ such that any position chosen in this box is appropriate. Obviously, the wider the box, the better.

Finally, our problem is defined as a set of constraints $c_i(x, y)$ $(1 \leq i \leq m)$, with $c_i(x, y)$ iff

$$\exists (a_i, b_i, d_i) \in (\mathbf{a}_i \times \mathbf{b}_i \times \mathbf{d}_i)\ (x - a_i)^2 + (y - b_i)^2 = d_i^2.$$

A solution of our problem is a tuple (x, y) such that for all $i \in [1..n]$, $c_i(x, y)$ is true and our goal is to build a so-called *inner box* $\mathbf{x} \times \mathbf{y}$, in which each point (x, y) is a solution [Ward *et al.*, 1989]. Classical interval analysis and constraint programming over the reals provide well-known algorithms for handling systems of equations with continuum of solutions [Benhamou and Goualard, 2000; Silaghi *et al.*, 2001; Vu *et al.*, 2002]. Nonetheless, they are not adapted for building inner boxes when the system involves existentially quantified parameters (especially when the system is not square

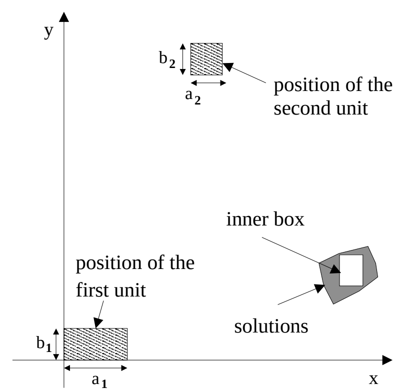

Figure 1: The relay positioning problem

w.r.t. the parameters). Some techniques either based on modal intervals [Herrero *et al.*, 2005], or Newton-like existence theorems [Goldsztejn, 2006] can detect inner boxes in presence of parameters, but one needs to enforce a whole branch-and-bound process to get an answer. Such a process is heavy and leads to disastrous computation time as the dimension n increases (merely because it tries to describe a (n-1)-dimension frontier with very small boxes). Worse, it is never sure that an inner box will be returned at the end.

We propose in this paper an original method for building an inner box around an initial solution of the parameter-free problem. This method starts with a degenerate box (a box reduced to a point, that can be obtained using a least-square method, for example) and tries successively to enlarge the dimensions of the box, while proving that the current box remains an inner box. Domain extension has already been achieved in case of parameter-free inequalities by defining an *univariate* extrema function and computing its left most and right most solutions of a selected variable, using a Newton like method [Collavizza *et al.*, 1999]. Our new extension algorithm works for parametric equations, thus subsuming inequalities and addressing more situations. It essentially extends one variable at a time and the resulting box depends on the order in which variables are selected. First of all, the functions we can handle are *arithmetical* functions.

Definition 1.1 (Arithmetical function) *f is said to be an arithmetical function, if the formal expression $f(x)$ matches the following recursive definition:*

- $f(x) = x_i$, *with* $i \in [1..n]$.
- $f(x) = c$, *where c is a constant in* $\mathbb{R}$.
- $f(x) = \phi(g(x))$, *where g is an arithmetical function, and ϕ is a "basic" function such as sqr, $sqrt$, sin, ...*
- $f(x) = g(x) \star h(x)$ *where g and h are arithmetical functions, and $\star$ is a binary operator in* $\{+, -, \times, /\}$.

The keystone of our domain extension is the *generalized projection* which will be introduced in the theoretical context of *modal interval analysis*. Informally, our method uses an inner box characterized by a generalized inclusion as $f(\mathbf{x}) \subseteq [2, -1]$. We know that, as long as $f(\mathbf{x}) \subseteq [0, 0]$, $\mathbf{x}$ is an inner box. Hence, we "enlarge" $\mathbf{x}$ as much as we can by considering a right-hand side "enlarged" to $[0, 0]$. We propagate this enlargement through the syntactic tree of f down to the leaf representing x.

2 Modal interval analysis

The theory of modal intervals has been developed by Spanish researchers since the 1980's [Gardeñes *et al.*, 1985; 2001]. It is a nice framework to deal with quantifiers in interval computations.

A simpler widely-adopted formulation of this theory has recently been proposed [Goldsztejn, 2005], and our next outline of modal intervals shall conform to this proposal.

First, let us define the general situation. Given a function f of variables $x = (x_1, \ldots, x_n)^T$ and a set of parameters $v = (v_1, \ldots, v_p)^T$, the solution set under study is

$$\{x \in \mathbb{R}^n \mid (\exists v \in \mathbf{v})\ f(x, v) = 0\}.$$

With classical interval arithmetics, evaluating a real-valued component f_i with interval vector operands $\mathbf{x}$ and $\mathbf{v}$ yields an interval $\mathbf{z}$ satisfying

$$(\forall x \in \mathbf{x})(\forall v \in \mathbf{v})(\exists z \in \mathbf{z}) \mid z = f(x, v).$$

Such a relation is not adequate for detecting an inner box. One would rather look for an interval $\mathbf{z}$ satisfying

$$(\forall z \in \mathbf{z})(\forall x \in \mathbf{x})(\exists v \in \mathbf{v}) \mid z = f(x, v),$$

thus $0 \in \mathbf{z}$ implies that $\mathbf{x}$ is an inner box.

Modal intervals analysis is an efficient tool for handling expressions built from intervals with associated quantifiers, paying special attention to the semantics behind the expression. The underlying structure of this theory is an extended set of intervals, called *generalized intervals*.

2.1 Generalized intervals

A **generalized interval** [Kaucher, 1980; Shary, 2002] is any pair $[a, b]$ of reals, without imposing $a \leq b$.

Here are some examples of generalized intervals: $[0, 1]$, $[1, 0]$, $[-1, 1]$, $[1, -1]$, $[0, 0]$. If $\mathbf{x}$ is a generalized interval $[a, b]$, we denote $\underline{\mathbf{x}} := a$ and $\overline{\mathbf{x}} := b$.

$\mathbb{KR}$ stands for the set of generalized intervals. It can be split into two subsets: the set $\mathbb{IR}$ of so-called **proper** intervals, those whose bounds are in increasing order, and the set of **improper** intervals, those whose bounds are in strictly decreasing order. Hence, $\mathbf{x}$ is proper if $\underline{\mathbf{x}} \leq \overline{\mathbf{x}}$ and $\mathbf{x}$ is improper if $\underline{\mathbf{x}} > \overline{\mathbf{x}}$. It is convenient to swap the endpoints of a generalized interval. For this end, the **dual** operator has been introduced :

$$\text{dual}\,(\mathbf{x}) := [\overline{\mathbf{x}}, \underline{\mathbf{x}}]$$

The **pro** operator for an interval is also defined to refer to its underlying set of reals (once the endpoints have been reordered properly).

$$\text{pro}\,(\mathbf{x}) := \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in \mathbb{IR} \\ (\text{dual } \mathbf{x}) & \text{otherwise} \end{cases}$$

Finally, $\mathbb{KR}$ is equipped with the following inclusion order:

$$\mathbf{x} \subseteq \mathbf{y} \iff \underline{\mathbf{x}} \geq \underline{\mathbf{y}} \wedge \overline{\mathbf{x}} \leq \overline{\mathbf{y}}. \tag{1}$$

E.g., $[2, -4] \subseteq [1, -3] \subseteq [0, 0] \subseteq [-3, 1] \subseteq [-4, 2]$.

$\mathbb{KR}$ is a complete lattice with respect to this inclusion. The *meet* and *join* of two intervals are respectively

$$\mathbf{x} \bigwedge \mathbf{y} = \max\{\mathbf{z} \mid \mathbf{z} \subseteq \mathbf{x} \wedge \mathbf{z} \subseteq \mathbf{y}\} = [\max\{\underline{\mathbf{x}}, \underline{\mathbf{y}}\}, \min\{\overline{\mathbf{x}}, \overline{\mathbf{y}}\}]$$

$$\mathbf{x} \bigvee \mathbf{y} = \min\{\mathbf{z} \mid \mathbf{x} \subseteq \mathbf{z} \wedge \mathbf{y} \subseteq \mathbf{z}\} = [\min\{\underline{\mathbf{x}}, \underline{\mathbf{y}}\}, \max\{\overline{\mathbf{x}}, \overline{\mathbf{y}}\}]$$

A generalized interval arithmetic is defined in [Kaucher, 1980]. Every binary operator and basic function (see Definition 1.1) is defined is such a way that it extends its counterpart in classical interval arithmetic. E.g., the addition in $\mathbb{KR}$ is :

$$\mathbf{x} + \mathbf{y} := [\underline{\mathbf{x}} + \underline{\mathbf{y}}, \overline{\mathbf{x}} + \overline{\mathbf{y}}],$$

so that $[1, 2] + [3, 5] = [4, 7]$ matches the result of classical interval addition, and $[1, 2] + [5, 3] = [6, 4]$. This extended arithmetic keeps the fundamental property of inclusion isotonicity (with the inclusion order (1)). Furthermore, $\mathbb{KR}$ is a group for addition and multiplication of zero-free intervals. The opposite of $\mathbf{x}$ is $-(\text{dual } \mathbf{x})$, the inverse is $(1/(\text{dual } \mathbf{x}))$. E.g., $[-1, 2] + [1, -2] = [0, 0]$, $\quad [1, 4] \times [1, 0.25] = [1, 1]$.

2.2 Main theorem

By chaining the basic arithmetic operators and functions, one can evaluate any expression with generalized intervals arguments. The theory of modal intervals has provided the following important interpretation [Gardeñes *et al.*, 2001; Goldsztejn, 2005]:

Proposition 2.1 *Let $\phi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ such that each component of v has only one occurrence in $\phi(x, v)$. Let $\mathbf{x} \in \mathbb{IR}^n$, $\mathbf{v} \in \mathbb{IR}^p$ and $\mathbf{z} := f(\mathbf{x}, \text{dual}(\mathbf{v}))$. If $\mathbf{z}$ is improper then $\big(\forall z \in \text{pro}(\mathbf{z})\big)\big(\forall x \in \mathbf{x}\big)\big(\exists v \in \mathbf{v}\big)\ z = \phi(x, v)$.*

Up to now, this proposition was mainly used as a *test* for inner boxes (e.g., [Grandón and Goldsztejn, 2006]). We detail this test in the next subsection.

2.3 Inner box test

Consider a set of constraints c_i ($1 \le i \le m$), each constraint being a parametric equation $f_i(x, v) = 0$ with $f_i : \mathbb{R}^n \times \mathbb{R}^p$. Assume that every component v_j ($1 \le j \le p$) only appears once in the whole system. To check if a given box $\mathbf{x}$ is inner, evaluate $f(\mathbf{x}, \text{dual}(\mathbf{v}))$. The result is a vector $\mathbf{z} \in \mathbb{KR}^m$. If $\mathbf{z} \subseteq 0$ then $\mathbf{x}$ is an inner box. Indeed, for all $i \in [1..m]$,

$$\big(\forall z_i \in \text{pro}(\mathbf{z}_i)\big)\big(\forall x \in \mathbf{x}\big)\big(\exists v \in \mathbf{v}\big)\ z_i = f_i(x, v).$$

Since $\mathbf{z}_i \subseteq 0 \iff 0 \in \text{pro}(\mathbf{z}_i)$ then

$$\big(\forall x \in \mathbf{x}\big)\big(\exists v \in \mathbf{v}\big)\ f_i(x, v) = 0.$$

Let us denote v_i the vector of parameters involved in c_i. Then $\big(\forall x \in \mathbf{x}\big)\big(\exists v_i \in \mathbf{v}_i\big)\ f_i(x, v_i) = 0$ is true for all $i \in [1..m]$, and this implies $\big(\forall x \in \mathbf{x}\big)\big(\exists v \in \mathbf{v}\big)\ f(x, v) = 0$.

As a new result, we will show that Proposition 2.1 can also be used as a constructive tool for inner boxes. Our technique combines this modal interval analysis result with a constraint programming concept called *projection*. Next section introduces the latter and expounds our contribution.

3 A Generalized Interval Projection

Let us first consider a real-valued arithmetical function $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$. We split variables into $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$, while $v \in \mathbb{R}^p$ is the vector of parameters. Thus, with no loss of generality, we shall write $f(x, y, v)$.

This section gives a technique to enlarge the domain of a variable that has only one occurrence[1] in the expression of the function[2], with given domains for other variables and parameters. So we assume x has only one occurrence in f, and fix once for all $\mathbf{y} \in \mathbb{IR}^{n-1}$ and $\mathbf{v} \in \mathbb{IR}^p$.

This technique handles $\mathbf{x}$ (the domain of x) as a variable and tries to find a solution in $\mathbb{KR}$ to some interval relation. We work at the interval level, which must be sharply distinguished from the usual standpoint of interval analysis: Instead of solving an equation of real variable/parameters and using intervals as a way to represent an infinite number of values, we solve an equation of interval variable and look for one interval solution.

To be applied, this technique requires that the variable $\mathbf{x}$ has a domain, i.e., a lower bound and an upper bound, w.r.t. the inclusion order defined by (1). So there must be intervals $\mathbf{x}_l$ and $\mathbf{x}_u$ such that $\mathbf{x}_l \times \mathbf{y}$ is the initial inner box we want to enlarge, and $\mathbf{x}_u$ is the domain of all possible values for x. Most of the time, it is easy to provide such an upper bound. Both bounds are proper. We can finally write

$$\mathbf{x}_l \subseteq \mathbf{x} \subseteq \mathbf{x}_u. \tag{2}$$

Our goal is to find a maximal interval $\mathbf{x} \in \mathbb{KR}$ (w.r.t. the inclusion defined by (1)) such that,

$$\mathbf{x} \text{ satisfies (2)} \quad \text{and} \quad f(\mathbf{x}, \mathbf{y}, (\text{dual } \mathbf{v})) \subseteq [0, 0],$$

i.e., such that $\mathbf{x}$ both satisfies the domain constraint and the inner test. If f is linear, some methods already tackle this problem [Markov *et al.*, 1996; Shary, 1996; 2002; Sainz *et al.*, 2002]. Consider now the (slightly) more general problem of finding a maximal $\mathbf{x}$ such that

$$\mathbf{x} \text{ satisfies (2)} \quad \text{and} \quad f(\mathbf{x}, \mathbf{y}, (\text{dual } \mathbf{v})) \subseteq \mathbf{z}, \tag{3}$$

with $\mathbf{z} \in \mathbb{KR}$ such that

$$f(\mathbf{x}_l, \mathbf{y}, (\text{dual } \mathbf{v})) \subseteq \mathbf{z} \subseteq f(\mathbf{x}_u, \mathbf{y}, (\text{dual } \mathbf{v})). \tag{4}$$

Notice that a maximal interval satisfying (3-4) is not necessarily a maximal inner extension of $\mathbf{x}_l$ in $\mathbf{x}_u$.

Using Definition 1.1, we can recursively solve (3) by isolating the subexpression containing x and applying one of the three "elementary" projections detailed below.

3.1 Overview

The recursion consists in reducing (3-4) to a simpler relation

$$\mathbf{x} \text{ satisfies (2)} \quad \text{and} \quad g(\mathbf{x}, \mathbf{y}, (\text{dual } \mathbf{v})) \subseteq \mathbf{z}', \tag{5}$$

where g is a subexpression of f, and $\mathbf{z}'$ satisfies

$$g(\mathbf{x}_l, \mathbf{y}, (\text{dual } \mathbf{v})) \subseteq \mathbf{z}' \subseteq g(\mathbf{x}_u, \mathbf{y}, (\text{dual } \mathbf{v})). \tag{6}$$

Relation (5-6) must be a sufficient condition to (3-4) in the sense that a maximal $\mathbf{x} \in \mathbb{KR}$ satisfying (5-6) must also be a maximal $\mathbf{x} \in \mathbb{KR}$ satisfying (3-4).

Given f, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{v}$ and $\mathbf{z}$, we detail now how to compute an appropriate $\mathbf{z}'$, dealing with three different cases. These cases are related to the syntactic decomposition of f given by Definition 1.1. The base case is straightforward. The other cases lie on three concepts: *theoretical projection*, *selection* and *filtering*.

3.2 Base case ($f(x, y, v) = x$)

By hypothesis, (4) holds, i.e., $\mathbf{x}_l \subseteq \mathbf{z} \subseteq \mathbf{x}_u$. Hence, a maximal $\mathbf{x}$ such that $\mathbf{x}$ satisfies (2) and $(\mathbf{x} \subseteq \mathbf{z})$ is $\mathbf{z}$ itself.

3.3 Basic function ($f(x, y, v) = \phi(g(x, y, v))$)

• Theoretical projection

For clarity, we replace $g(\mathbf{x}, \mathbf{y}, (\text{dual } \mathbf{v})$ by the symbol $\mathbf{g}$. Since every basic function ϕ is piecewise strictly monotonic,

[1] This is a limitation due to the *dependency* problem of interval arithmetic. It can be solved by applying a fixed point algorithm over the multi-ocurrence variable, but this is out of scope of this article.

[2] This presentation is done for one constraint. In presence of several constraints, the same operation is performed for each constraint and the intersection of the obtained intervals is returned.

hence piecewise invertible, for any $\mathbf{z} \in \mathbb{KR}$, a disjunction of inclusions

$$(\mathbf{g} \subseteq \mathbf{z}_1) \text{ or } (\mathbf{g} \subseteq \mathbf{z}_2) \text{ or } \ldots$$

can formally be derived from $\phi(\mathbf{g}) \subseteq \mathbf{z}$, regardless of condition (2). For example,

$$\exp(\mathbf{g}) \subseteq [1,2] \iff \mathbf{g} \subseteq [0, \log(2)],$$

$$\mathbf{g}^2 \subseteq [4,0] \iff \mathbf{g} \subseteq [2,0] \text{ or } \mathbf{g} \subseteq [0,-2].$$

Notice that if $\phi = sqr$, $\mathrm{pro}\,(\mathbf{z})$ cannot include negative values, so that the square root is always well defined. Indeed, by hypothesis, (4) holds. If $\mathbf{z}$ is proper, then $\mathbf{z} \subseteq g(\mathbf{x}_u, \mathbf{y}, (\mathrm{dual}\ \mathbf{v}))^2$ and $g(\mathbf{x}_u, \mathbf{y}, (\mathrm{dual}\ \mathbf{v}))^2 \geq 0$ implies $\mathrm{pro}\,(\mathbf{z}) \geq 0$. Otherwise, $g(\mathbf{x}_l, \mathbf{y}, (\mathrm{dual}\ \mathbf{v}))^2 \subseteq \mathbf{z}$, i.e., $\mathrm{pro}\,(\mathbf{z}) \subseteq \mathrm{pro}\,(g(\mathbf{x}_l, \mathbf{y}, (\mathrm{dual}\ \mathbf{v}))^2)$ which again implies $\mathrm{pro}\,(\mathbf{z}) \geq 0$. This symmetry in the domain of $sqrt$ and the image of sqr is obviously valid for every basic function.

As soon as ϕ is trigonometric, the disjunction includes an infinity of terms (which justifies the "theoretical" qualifier):

$$\cos(\mathbf{g}) \subseteq [0.5, 1] \iff \mathbf{g} \subseteq [\pi/3, \pi/2] \text{ or } \ldots$$

All intervals in the (possibly infinite) sequence share both the same proper/improper nature and the same diameter. Furthermore, either their proper projections are all disjoint ($i \neq j \Longrightarrow \mathrm{pro}\,(\mathbf{z}_i) \cap \mathrm{pro}\,(\mathbf{z}_j) = \emptyset$), either they all intersect. They cannot however overlap more than a bound. One may wonder if two overlapping intervals $\mathbf{g}_1$ and $\mathbf{g}_2$ can be merged, i.e., if the condition $(\mathbf{g} \subseteq \mathbf{z}_1)$ or $(\mathbf{g} \subseteq \mathbf{z}_2)$ can be replaced by $\mathbf{g} \subseteq (\mathbf{z}_1 \vee \mathbf{z}_2)$. This is not allowed since $\mathbf{g} \subseteq (\mathbf{z}_1 \vee \mathbf{z}_2)$ is only a necessary condition (as counter-example, $\mathbf{g} := [-1, 1]$ satisfies $\mathbf{g} \subseteq [0,2] \vee [-2,0] = [-2,2]$ but neither satisfies $\mathbf{g} \subseteq [0,2]$ nor $\mathbf{g} \subseteq [-2,0]$). In contrast, $\mathbf{g} \subseteq (\mathbf{z}_1 \wedge \mathbf{z}_2)$ is a sufficient but stronger condition, and maximality is lost (no solution can even be found). Thus, no merging of any kind can be done. Summing up, solving (3) boils down to solving

$$\mathbf{x} \text{ satisfies (2)} \quad \text{and} \quad g(\mathbf{x}, \mathbf{y}, (\mathrm{dual}\ \mathbf{v}))) \subseteq \mathbf{z}_j \qquad (7)$$

for one $\mathbf{z}_j$ in the sequence. We can now avail ourselves of the constraint on the domain of $\mathbf{x}$ to *select* and *filter* a feasible interval in this sequence. *Selection* means that we pick an interval $\mathbf{z}_j$ such that a solution $\mathbf{x}$ of (7) exists. *Filtering* means that we find the largest $\mathbf{z}' \subseteq \mathbf{z}_j$ such that (6) is satisfied.

• Selection

Relation (2) allows us to keep only a finite number of z_j in the theoretical projection. By inclusion isotonicity of Kaucher arithmetic, $\mathbf{x}_l \subseteq \mathbf{x}$ implies $g(\mathbf{x}_l, \mathbf{y}, \mathrm{dual}\ \mathbf{v}) \subseteq g(\mathbf{x}, \mathbf{y}, \mathrm{dual}\ \mathbf{v})$. So we can detect whether $\mathbf{z}_j$ $(j = 1, 2, \ldots)$ is feasible or not by checking $g(\mathbf{x}_l, \mathbf{y}, (\mathrm{dual}\ \mathbf{v})) \subseteq \mathbf{z}_j$. The number of feasible $\mathbf{z}_j$ resulting from this test is necessarily finite (see Example 3.2). We can pick any one of them.

Example 3.1 *Consider* $f(x,y,v) = (x+v)^2$, $\mathbf{x}_l = [-1,-1]$, $\mathbf{x}_u = [-2,3]$, $\mathbf{v} = [-1,2]$ *and* $\mathbf{z} = [4,1]$. *Then, we have* $\phi = sqr$, $g(x,y,v) = x + v$ *and*

$$(\mathbf{x} + (\mathrm{dual}\ \mathbf{v}))^2 \subseteq [4,1] \iff \begin{cases} \mathbf{x} + (\mathrm{dual}\ \mathbf{v}) \subseteq [2,1] \text{ or} \\ \mathbf{x} + (\mathrm{dual}\ \mathbf{v}) \subseteq [-1,-2] \end{cases}$$

But since $\mathbf{x}_l + (\mathrm{dual}\ \mathbf{v}) = [1,-2]$, $[2,1]$ *is not feasible (because* $[1,-2] \not\subseteq [2,1]$*) whereas* $[-1,-2]$ *is feasible (*$[1,-2] \subseteq [-1,-2]$*).*

For the sake of simplicity, we performed in the last example theoretical projection and selection consecutively, as two separate steps. With trigonometric functions, this is not possible as the number of theoretical projections is infinite. So, we rather use selection as a pre-selecting process. This is illustrated on the next example.

Example 3.2 *Consider* $f(x,y,v) = cos(x+v)$, $\mathbf{x}_l = [6,6]$, $\mathbf{x}_u = [5,9]$, $\mathbf{v} = [-1,1]$ *and. Then, we have* $\phi = cos$ *and* $g(x,y,v) = x + v$. *We first compute*

$$\mathbf{g}_l := \mathbf{x}_l + (\mathrm{dual}\ \mathbf{v}) = [7,5],$$

It follows that $\mathrm{pro}\,(\mathbf{g}_l) \subseteq [5,7]$, *which restricts the projection of cosinus to two half periods,* $[\pi, 2\pi]$ *and* $[2\pi, 3\pi]$*:*

$$\mathbf{x} + (\mathrm{dual}\ \mathbf{v}) \subseteq 2\pi + arccos([0.7, 0.8]) = [6.93, 7.08]$$

or

$$\mathbf{x} + (\mathrm{dual}\ \mathbf{v}) \subseteq 2\pi - arccos([0.7, 0.8]) = [5.49, 5.64].$$

• Filtering

Once $\mathbf{z}_j$ was proven to be feasible, relation (2) can be used to make $\mathbf{z}_j$ smaller and fulfill (6). Indeed, $\mathbf{x} \subseteq \mathbf{x}_u$ implies $g(\mathbf{x}, \mathbf{y}, \mathrm{dual}\ \mathbf{v}) \subseteq g(\mathbf{x}_u, \mathbf{y}, \mathrm{dual}\ \mathbf{v})$. Hence we can substitute $\mathbf{z}_j$ by $\mathbf{z}_j \wedge g(\mathbf{x}_u, \mathbf{y}, (\mathrm{dual}\ \mathbf{v}))$.

Example 3.3 *In Example 3.1, we found out that interval* $[-1,-2]$ *was feasible. But as* $\mathbf{x}_u + (\mathrm{dual}\ \mathbf{v}) = [0,2]$, *we must actually have* $\mathbf{x} + (\mathrm{dual}\ \mathbf{v}) \subseteq [0,2] \wedge [-1,-2] = [0,-2]$. *This condition is indeed stronger.*

3.4 Binary Operator ($f(x,y,v) = g(x,y,v) \star h(y,v)$)

Put $\mathbf{w} := h(\mathbf{y}, (\mathrm{dual}\ \mathbf{v}))$ and consider first the addition. The inclusion $f(\mathbf{x}, \mathbf{y}, (\mathrm{dual}\ \mathbf{v})) \subseteq \mathbf{z}$ turns to

$$g(\mathbf{x}, \mathbf{y}, (\mathrm{dual}\ \mathbf{v})) + \mathbf{w} \subseteq \mathbf{z}.$$

By adding $-\,\mathrm{dual}\,(\mathbf{w})$ to each side of the latter, we get

$$g(\mathbf{x}, \mathbf{y}, (\mathrm{dual}\ \mathbf{v})) \subseteq \mathbf{z} - (\mathrm{dual}\ \mathbf{w})$$

thanks to the group property of Kaucher arithmetic. Filtering can apply here to narrow (or possibly empty) $\mathbf{z} - (\mathrm{dual}\ \mathbf{w})$. The same idea applies to subtraction and division (by respectively adding and multiplying $\mathbf{z}$ by $(\mathrm{dual}\ \mathbf{w})$).

Multiplication however requires some precaution. If $0 \notin \mathrm{pro}\,(\mathbf{w})$, then we can again divide $\mathbf{z}$ by $(\mathrm{dual}\ \mathbf{w})$. But if $0 \in \mathrm{pro}\,(\mathbf{w})$, because Kaucher arithmetic does not handle infinite bounds we need to hand-craft a special division. Similar extensions of Kaucher's division are proposed in [Popova, 1994; Goldsztejn, 2005]. A maximal $\mathbf{g}$ satisfying $\mathbf{g} \times \mathbf{w} \subseteq \mathbf{z}$ is obtained with the next table.

	$\mathbf{z} > \mathbf{0}$	$\mathbf{z} < \mathbf{0}$	$\mathbf{0} \subseteq \mathbf{z}$	$\mathbf{z} \subset \mathbf{0}$
$\mathbf{0} \subseteq \mathbf{w}$	1	2	3	4
$\mathbf{w} \subset \mathbf{0}$	5	6	7	8

1. $\mathbf{g} = \emptyset$ (no solution)
2. $\mathbf{g} = \emptyset$ (no solution)
3. $\mathbf{g} \subseteq [\max\{\underline{\mathbf{z}}/\overline{\mathbf{w}}, \overline{\mathbf{z}}/\underline{\mathbf{w}}\}, \min\{\underline{\mathbf{z}}/\underline{\mathbf{w}}, \overline{\mathbf{z}}/\overline{\mathbf{w}}\}]$
4. $\mathbf{g} = \emptyset$ (no solution)
5. $\mathbf{g} \subseteq [-\infty, \underline{\mathbf{z}}/\overline{\mathbf{w}}]$ or $\mathbf{g} \subseteq [\underline{\mathbf{z}}/\underline{\mathbf{w}}, +\infty]$

6. $\mathbf{g} \subseteq [-\infty, \overline{\mathbf{z}}/\underline{\mathbf{w}}]$ or $\mathbf{g} \subseteq [\overline{\mathbf{z}}/\overline{\mathbf{w}}, +\infty]$
7. $\mathbf{g} \subseteq [-\infty, +\infty]$
8. $\mathbf{g} \subseteq [-\infty, \min\{\underline{\mathbf{z}}/\overline{\mathbf{w}}, \overline{\mathbf{z}}/\underline{\mathbf{w}}\}]$ or $[\max\{\underline{\mathbf{z}}/\underline{\mathbf{w}}, \overline{\mathbf{z}}/\overline{\mathbf{w}}\}, +\infty]$

Applying filtering on $\mathbf{g}$ (a consistent extension of (1) to intervals with infinite bounds is easy) immediately removes infinite bounds since $\mathbf{g}$ is necessarily proper. Hence, infinite bounds are not propagated to subsequent computations (which would have led to undefined results). They only are a convenient way to represent arbitrarily large values when enforcing filtering.

Example 3.4 *Consider* $f(x, y, v) = x \times v$, $\mathbf{x}_l = [-1, 1]$, $\mathbf{x}_u = [-3, 3]$, $\mathbf{v} = [-1, 2]$ *and* $\mathbf{z} = [-2, 6]$. *Then, we have* $\star = \times$, $g(x, y, v) = x$. *Thanks to the table, we get*

$$\mathbf{x} \subseteq [-\infty, -1] \textit{ or } \mathbf{x} \subseteq [1, +\infty].$$

Both contain $\mathbf{x}_l$, *hence are feasible. Applying "meet" operator with* $\mathbf{x}_u$ *yields* $\mathbf{x} \subseteq [-3, -1]$ *or* $\mathbf{x} \subseteq [1, 3]$.

Remark 1 *We have seen that the constraint on the domain is crucial in presence of trigonometric functions or multiplication with* 0 *in operands. In the other cases, by removing domain constraint (i.e., condition* (2)*), it can be easily proven that a maximal* $\mathbf{x}$ *satisfying*

$$f(\mathbf{x}, \mathbf{y}, (\text{dual } \mathbf{v})) \subseteq [0, 0]$$

also satisfies $\quad f(\mathbf{x}, \mathbf{y}, (\text{dual } \mathbf{v})) = [0, 0]$.

It is worth mentioning that functions need not be decomposed formally into subexpressions: projections are directly performed by an automatic projection algorithm [Benhamou *et al.*, 1999], similar to automatic differentiation.

We detail now a trace of our extension algorithm.

4 Trace

We instantiate our relay example with 4 units. According to the problem in section 1, the set of constraints are:

$$\exists (a_i, b_i, d_i) \in (\mathbf{a}_i \times \mathbf{b}_i \times \mathbf{d}_i)\ (x - a_i)^2 + (y - b_i)^2 = d_i^2,$$

and the domains of the parameters are:

$\mathbf{a}_1 = [0, 2]$	$\mathbf{b}_1 = [0, 1]$	$\mathbf{d}_1 = [1, 8]$
$\mathbf{a}_2 = [4, 5]$	$\mathbf{b}_2 = [9, 10]$	$\mathbf{d}_2 = [1, 8]$
$\mathbf{a}_3 = [13, 15]$	$\mathbf{b}_3 = [-11, -10]$	$\mathbf{d}_3 = [1, 14]$
$\mathbf{a}_4 = [16, 17]$	$\mathbf{b}_4 = [5, 7]$	$\mathbf{d}_4 = [1, 8]$

A least-square solution obtained by fixing each parameter to the midpoint of its domain is $(\tilde{x} = 9.04286, \tilde{y} = 2.6494)$. We first check that this solution can be taken as the starting point of our domain extension. We compute for all i,

$$(\mathbf{x} - \text{dual}\,(\mathbf{a}_i))^2 + (\mathbf{y} - \text{dual}\,(\mathbf{b}_i))^2 - \text{dual}\,(\mathbf{d}_i)^2$$

with $\mathbf{x} = [\tilde{x}, \tilde{x}]$ and $\mathbf{y} = [\tilde{y}, \tilde{y}]$. We get the following image vector :

$$[87.8, -11.7], [78.5, -7.3], [220.8, -20.3], [81.2, -10].$$

As this vector is included in 0, then the initial degenerate box $\mathbf{x} \times \mathbf{y}$ is an inner box. We can now decide that the position (x, y) should not be out of a bounding box $\mathbf{x}_u \times \mathbf{y}_u = [5, 15] \times [0, 20]$. The extension of $\mathbf{x}$ can start.

We detail the projection of c_1 over x. Our goal is to find the biggest $\mathbf{x}$ ($\tilde{\mathbf{x}} \subseteq \mathbf{x} \subseteq \mathbf{x}_u$) such that

$$\boxed{(\mathbf{x} - (\text{dual } \mathbf{a}_1))^2 + (\mathbf{y} - (\text{dual } \mathbf{b}_1))^2 - (\text{dual } \mathbf{d}_1)^2 \subseteq 0}$$

• Apply Case 3.
Compute $\mathbf{w} := (\mathbf{y} - (\text{dual } \mathbf{b}_1))^2 - (\text{dual } \mathbf{d}_1)^2$. We get $\mathbf{w} = (2.6494 - [1, 0])^2 - [8, 1]^2 = [6.02, -61.28]$. Then,

$$(\mathbf{x} - (\text{dual } \mathbf{a}_1))^2 + \mathbf{w} \subseteq 0 \Rightarrow (\mathbf{x} - (\text{dual } \mathbf{a}_1))^2 \subseteq -(\text{dual } \mathbf{w})$$

Finally, $(\mathbf{x} - (\text{dual } \mathbf{a}_1))^2 \subseteq [-6.02, 61.28]$.

We apply domain restriction. We first compute $(\tilde{x} - (\text{dual } \mathbf{a}_1))^2 = [81.77, 49.60]$ and check that $[81.77, 49.60] \subseteq [-6.02, 61.28]$. We also compute $\mathbf{x}_u - (\text{dual } \mathbf{a}_1))^2 = [25, 169]$ and filter $[-6.02, 61.28]$ to $[25, 61.28]$.

$$\boxed{(\mathbf{x} - (\text{dual } \mathbf{a}_1))^2 \subseteq [25, 61.28]}$$

• Apply Case 2.

$$\begin{array}{c}(\mathbf{x} - (\text{dual } \mathbf{a}_1)) \subseteq [5, \sqrt{61.28}] = [5, 7.82] \\ \vee \\ (\mathbf{x} - (\text{dual } \mathbf{a}_1)) \subseteq [-\sqrt{61.28}, -5] = [-7.82, -5]\end{array}$$

But $(\mathbf{x}_l - (\text{dual } \mathbf{a}_1)) = [9.04, 7.04]$ and $(\mathbf{x}_u - (\text{dual } \mathbf{a}_1)) = [5, 13]$. So, by domain restriction, $[-7.82, -5]$ is discarded, and $[5, 7.82]$ is left intact.

$$\boxed{\mathbf{x} - (\text{dual } \mathbf{a}_1) \subseteq [5, 7.82]}$$

• Apply Case 3.
$\mathbf{x} \subseteq [5, 7.82] + \mathbf{a}_1 \iff \boxed{\mathbf{x} \subseteq [5, 9.82]}$.

• Apply Case 1: the answer is $[5, 9.82]$.

We perform a generalized projection to compute consistent extension of $\tilde{x}$ w.r.t the other constraints and get three other intervals: $\mathbf{x}_2 = [5, 9.86]$, $\mathbf{x}_3 = [7, 15]$ and $\mathbf{x}_4 = [8.36, 15]$. The intersection of the four intervals, $[8.36, 9.82]$, is inner w.r.t. the whole system (See Section 2.3). We can perform now a generalized projection to compute an extension over y of the new box $[8.36, 9.82] \times \tilde{y}$ and we get respectively for each constraint $\mathbf{y}_1 = [0, 2.6494]$, $\mathbf{y}_2 = [2.62, 16]$, $\mathbf{y}_3 = [0, 3.20]$ and $\mathbf{y}_4 = [2.64, 9.35]$. The intersection of these extensions is $[2.6494, 2.6494]$. The final inner box $[8.36, 9.82] \times [2.6494, 2.6494]$ is shown in Figure 2(a).

Let us roll back this extension. If we start domain extension over the variable y at first, and then, over x, we obtain another box, $[9.04286, 9.04286] \times [2.1, 3.42]$ shown in 2(b). We observe that the maximal extension obtained for the first variable we project over generally prevents the other variables from being extended. In order to obtain more balanced boxes, we introduce a heuristic of extension in two steps. First, we extend all variables but the last one to the middle point between the initial value and the bounds of the maximal extension. The last variable is extended to the maximal interval. For example, x will be extended to $[8.7, 9.43]$ instead of $[8.36, 9.82]$, and then y will be extended to $[2.35, 3.32]$. Second, we perform a maximal extension for all variables (if they can again be extended). Figures 2(c) and 2(d) show the

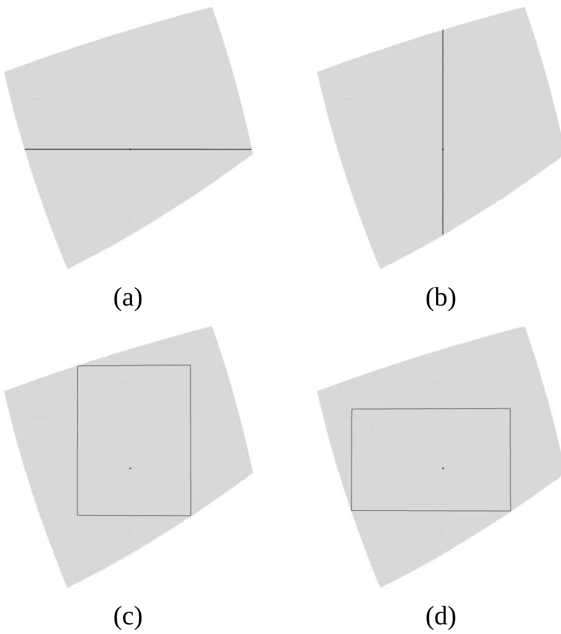

Figure 2: (a). Inner box with maximal extension of x and (b) with maximal extension of y. (c). First step of heuristic starting with x. (d). First step of heuristic starting with y.

results obtained with the first step of the heuristic, starting with variable x and variable y (boxes $[8.7, 9.43] \times [2.35, 3.32]$ and $[8.45, 9.48] \times [2.373, 3.039]$), respectively. Figure 2(c) shows a maximal inner box, while figure 2(d) can be extended again. Step two will extend y to a maximal interval, which is $[2.373, 3.237]$.

5 Conclusion

This paper provides a new method for extending consistent domains with parametric equations over the reals. The key point is the *generalized projection*, a new operator that combines a constraint programming concept with theoretical results from modal interval analysis. This projection can be computed in linear time w.r.t. the number of operators and functions involved in the equations. This makes our approach cheap and efficient. Furthermore, universally quantified parameters can also be included straightforwardly. Some limitations remain: parameters must occur once in the whole system, and variables cannot appear more than once in a given equation. Despite of these limitations, this is an original and promising approach to handle parametric equations, especially when existentially quantified parameters are involved (problems with uncertainties).

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